1. Relations between displacement and deformation. Geometric equations

1.1 Small deformations

Consider a deformable body in a Cartesian coordinate system and load it with a static system of external forces. Because the body is fixed (cannot move as a solid) due to these forces the initial configuration (geometry) of the body changes and the body points move to a new position - the body deforms.

Suppose we know the vector of displacement of body points (the FEM provides them on finite elements in approximative form)

$$\mathbf{u}(x,y,z) \equiv u_i(x,y,z) \equiv \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} \tag{1.1}$$

where the continuous functions u, v, and w are the displacement components of the general *xyz*-point in the direction of the coordinate axes. We are interested in the relationship between these functions and the searched functions, which will express the degree of deformation of the body at its points.

Let us denote the differential ("infinite" small) volume element *dxdydz* inside the unloaded body at the general point *xyz* (Fig. 1.1).

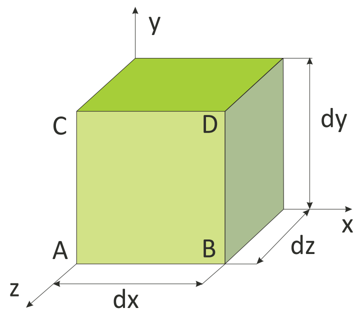

Fig. 1.1

Due to the load the element is displaced in space and deformed (due to its very small dimensions a simple type of strain related to the definition of stress is considered: its length dimensions change - in the direction of the coordinate axes stretches or shortens - and the right angles between its walls are violated) .

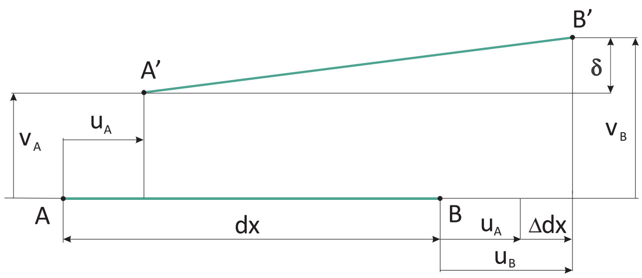

Fig. 1.2

Consider the element two points A and B on a parallel edge with the x-axis and spaced by *dx* (Fig. 1.2). The original position of points A and B will change as a result of the load. Their line moves, changes its length and rotates. For small deformations we neglect the effect of the rotation on the length of the line and for the degree of deformation (strain) of the element in the *x*- direction the relative change of the *x*-distance of the points is chosen

$$\varepsilon_x = \frac{\Delta dx}{dx} = \frac{u_B - u_A}{dx} \tag{1.2}$$

i.e. change in distance between points Δdx divided by the original distance *dx*. Of course, if the element moves only as a whole ($u_B = u_A$) in the x-axis direction, then $\varepsilon_x = 0$. As you can see, the strain is a signed dimensionless number (with the physical meaning of relative extension or truncation).

The displacement u_B can be determined from the development of the function *u* at point A to the truncated Taylor series. Because of the small deformations only the first two members are considered

$$u_B = u_A + \frac{\partial u}{\partial x} dx \tag{1.3}$$

which actually represents linearization of the function *u* at point A. By substituting (1.3) to (1.2) we get the relation for the calculation of the strain ε_x at points of the body

$$\varepsilon_x(x,y,z) = \frac{\partial u}{\partial x} \tag{1.4}$$

It is actually the gradient of the function $u(x,y,z)$.

By an analogous analysis of the deformation of the element in the *y* and *z* direction would be obtained

$$\varepsilon_y(x,y,z) = \frac{\partial v}{\partial y} \tag{1.5}$$

$$\varepsilon_z(x,y,z) = \frac{\partial w}{\partial z} \tag{1.6}$$

The functions ε_x, ε_y and ε_z, which we call *normal* strains, cause a change in the volume of the element but do not break the right angles between its walls, because they only shift the parallel walls of the element by a different value in the direction of their common normal.

The load of the body, however, also causes so-called *shear* strains γ_{xy}, γ_{yz}, γ_{zx} which disrupt the perpendicularity of the walls but do not alter the volume of the element. If, for example, point C (Fig. 1.1) shifts in the direction of the *x*-axis relative to point A by a value Δ, then according to Fig. 1.3

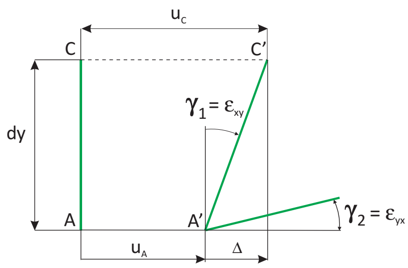

Fig. 1.3

by re-using the shortened Taylor series for the u_C we get

$$\Delta = u_C - u_A = u_A + \frac{\partial u}{\partial y}dy - u_A = \frac{\partial u}{\partial y}dy$$

In this way, the vertical wall of the element is tilted by an angle

$$\gamma_1 \approx tan\gamma_1 = \frac{\Delta}{dy} = \frac{\partial u}{\partial y}$$

Analogous for rotating the horizontal wall (Fig. 1.1) we have

$$\gamma_2 \approx tan\gamma_2 = \frac{\partial v}{\partial x}$$

and the shear deformation in the xy plane (total change in the right angle of the element in the xy plane) is

$$\gamma_{xy} = \gamma_1 + \gamma_2 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{1.7}$$

By cyclic interchange of variables we get shear deformations in the other two planes

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{1.8}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \tag{1.9}$$

In summary, these *linear* shear components at the general point of the body can be written into a (pseudo) vector

$$\{\boldsymbol{\varepsilon}\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} \end{Bmatrix} \tag{1.10}$$

The matrix notation of the symmetrical strain *tensor* considers the tensor shear deformation $\varepsilon_{ij} = \varepsilon_{ji}$ (Fig. 1.3) on each wall of the element (as a half value of the total shear deformation) and holds

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{x} & 0{,}5\gamma_{xy} & 0{,}5\gamma_{xz} \\ 0{,}5\gamma_{xy} & \varepsilon_{y} & 0{,}5\gamma_{yz} \\ 0{,}5\gamma_{xz} & 0{,}5\gamma_{yz} & \varepsilon_{z} \end{bmatrix} \tag{1.11}$$

where we get members from a simple index entry

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{1.12}$$

By the deformation formulation of the strength problem, where the displacement field is primarily unknown, the strain relations (1.10) clearly determine the deformation field. The inverse problem that occurs at the force formulations is more complex, because in the case of six deformation components the three displacement components can be unambiguously determined only with respect to the constraints that are contained in so-called *compatibility equations.*

1.2 Large strains

Linear geometric relations between displacement and strain components (1.10), resp. (1.12) at the material points of the deformed body are well suited for a large number of strength problems, since important materials used in engineering (metals, concrete, wood, glass, etc.) allow only small values of strains. E.g. if one-meter long rod of constant cross-section is stretched by 1 mm, it causes numerically small strain $\varepsilon_x = 0{,}001$. However, such strain in the steel rod causes a stress of about 200 MPa, which is at the limit of the yield stress of conventional steel, for several other less ductile materials it is already beyond the strength limit.

On the other hand, there are several materials (eg industrial rubber, some plastics) and several technological processes (eg different types of steel forming) where linear geometric equations do not fit and more precise relationships must be used in simulation calculations. Derivation of these relationships is not easy, there are several methods and applied measures of deformation, nor is their application in FEM. Here we will only present the geometrical way of deriving the components of the Green-Lagrange strain tensor which already gives some insight into nonlinear geometric relations, but it is necessary to say that the properties and possibilities of utilization of each measure of strain can only be more clearly explained together with its power partner - stress.

Returning to Fig. 1.2 and specify the change in the length of the connecting line of points A and B taking into account also the effect of different displacement of these points in the y-axis direction. We denoted this difference δ and by approximating the function around A by the shortened Taylor series we get

$$v_B = v_A + \frac{\partial v}{\partial x} dx$$

so

$$\delta = v_B - v_A = \frac{\partial v}{\partial x} dx$$

and the deformed length of the line is expressed by the Pythagorean theorem

$$A'B' = \sqrt{\left(dx + \frac{\partial u}{\partial x} dx\right)^2 + \left(\frac{\partial v}{\partial x} dx\right)^2} = dx\sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

We introduce a relative deformation (marked as for small deformations)

$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{A'B'}{dx} - 1 = \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2} - 1$$

The function of shape $\sqrt{1+x}$ can be decomposed into a series $1 + x/2 + x^2/8 + \ldots$ and the previous relationship after approximating the member with a square root by two series members is given by

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \tag{1.13}$$

Analogously, it can be shown that

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \tag{1.14}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} \tag{1.15}$$

It can be seen that when the deformations are sufficiently small the quadratic terms can be neglected and we get the deformation components derived for the linear case.

Previous relations can be completed by analogous terms for *z*-direction and we can write the *independent* components of Green-Lagrange strain tensor into vector

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \\ \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial w}{\partial x} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{2}\left(\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial x}\right)^2\right) \\ \dfrac{1}{2}\left(\left(\dfrac{\partial u}{\partial y}\right)^2 + \left(\dfrac{\partial v}{\partial y}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right) \\ \dfrac{1}{2}\left(\left(\dfrac{\partial u}{\partial z}\right)^2 + \left(\dfrac{\partial v}{\partial z}\right)^2 + \left(\dfrac{\partial w}{\partial z}\right)^2\right) \\ \left(\dfrac{\partial u}{\partial x}\right)\left(\dfrac{\partial u}{\partial y}\right) + \left(\dfrac{\partial v}{\partial x}\right)\left(\dfrac{\partial v}{\partial y}\right) + \left(\dfrac{\partial w}{\partial x}\right)\left(\dfrac{\partial w}{\partial y}\right) \\ \left(\dfrac{\partial u}{\partial y}\right)\left(\dfrac{\partial u}{\partial z}\right) + \left(\dfrac{\partial v}{\partial y}\right)\left(\dfrac{\partial v}{\partial z}\right) + \left(\dfrac{\partial w}{\partial y}\right)\left(\dfrac{\partial w}{\partial z}\right) \\ \left(\dfrac{\partial u}{\partial x}\right)\left(\dfrac{\partial u}{\partial z}\right) + \left(\dfrac{\partial v}{\partial x}\right)\left(\dfrac{\partial v}{\partial z}\right) + \left(\dfrac{\partial w}{\partial x}\right)\left(\dfrac{\partial w}{\partial z}\right) \end{bmatrix} \tag{1.16}$$

or in a brief index tensor notation

$$\varepsilon_{k\ell} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_\ell} + \frac{\partial u_\ell}{\partial x_k} + \frac{\partial u_m}{\partial x_k}\frac{\partial u_m}{\partial x_\ell}\right) \tag{1.17}$$

Remark 1: If a bar (rod, truss) with a length ℓ_0 parallel to the *x*-axis at the end (located at the beginning of the coordinate system) is fixed and stretched by $\Delta\ell$ at the other end, its length will change to $\ell = \ell_0 + \Delta\ell$ and the displacement function will be

$$u(x) = \frac{\Delta\ell}{\ell_0}x = \frac{\ell - \ell_0}{\ell_0}x$$

According to the relationship (1.4) valid for small deformations, the standard "engineering" strain of the bar is

$$\varepsilon_{ing} = \frac{\partial u}{\partial x} = \frac{\ell - \ell_0}{\ell_0}$$

From (1.13) we get the Green's strain

$$\varepsilon_G = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = \varepsilon_{ing} + \frac{1}{2}\varepsilon_{ing}^2 = \frac{\ell - \ell_0}{\ell_0} + \frac{1}{2}\left(\frac{\ell - \ell_0}{\ell_0}\right)^2 = \frac{1}{2}\frac{\ell^2 - \ell_0^2}{\ell_0^2} \tag{1.18}$$

For small deformations ($\varepsilon_{ing} << 1$) the difference between the two measures is negligible:

$$\varepsilon_{ing} = \frac{l - l_0}{l_0} = \frac{(l - l_0)(l + l_0)}{l_0(l + l_0)} = \frac{l^2 - l_0^2}{l_0(l_0 + \Delta l + l_0)} = \frac{l^2 - l_0^2}{l_0^2(2 + \varepsilon_{ing})} \approx \frac{l^2 - l_0^2}{2l_0^2} = \varepsilon_G$$

The Green-Lagrange strain tensor is independent of the rigid displacement and solid rotation of the body and is therefore often used in the FEM to solve problems with small deformations but large displacements and large rotations in the so-called corotational coordinate system.

Remark 2: Let's return to the figure from which we determined the components of the Green-Lagrange strain tensor:

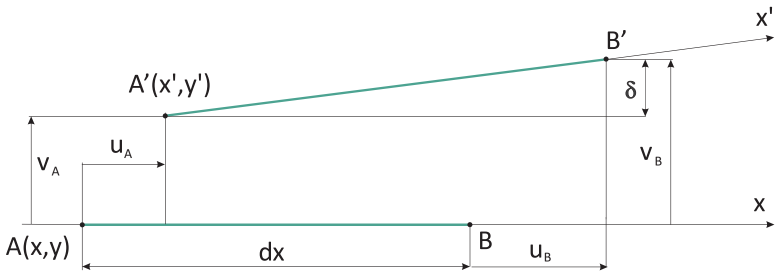

Fig. 1.4

The Green-Lagrange strain is, as already mentioned, expressed by the ratio

$$\varepsilon_x = \frac{A'B' - AB}{AB}$$

It is an expression analogous to small deformations but note that now the direction of the line *A'B'* and hence the direction ε_x is different from the direction *AB*, respectively *dx*. The component has a direction identical to the direction in which the segment *dx* is rotated under the load. If in the next loading step the line is only moved and rotated without changing the distance *A'B'* then the amount of deformation does not change. The components of the Green-Lagrange strain follow in their direction the rotation of the element (material particle) but their size is *independent* of this rotation (and of course the rigid displacement). This also valid for their partner (energy-bound) stress components of the second Piola-Kirchhoff stress (Chapter 5).